

# Algebraization of bundles on non-proper schemes.

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## 1 Introduction

This work is a contribution toward an algebraic understanding of the Uhlenbeck compactification. Recall, cf. [DK] that for a complex projective surface  $S$  the moduli space  $M_n$  of semistable vector bundles with fixed rank, determinant and  $c_2 = n$  is non-compact, but the union  $Uhl_n = \coprod_{s \geq 0} M_{n-s} \times \text{Sym}^s S$  can be given a topology of a compact space (since one deals with semistable bundles for  $s \gg 0$  the space  $M_{n-s}$  will be empty). We will call  $Uhl_n$  the Uhlenbeck moduli space although sometimes this name is reserved for the closure of  $M_n$  in  $Uhl_n$ .

Some time ago, see e.g. [Li], [BFG], [FGK], the Uhlenbeck moduli space started to appear in algebraic geometry and higher dimensional Langlands Program. For instance, it is a convenient tool for the study of higher versions of Hecke correspondences which modify a vector bundle on  $S$  (more generally, a principal bundle) along a divisor, obtaining a new bundle. For several reasons, we would like to have a definition of  $Uhl_n$  as a “functor”, i.e. we want to be able to describe in geometric terms the set of maps  $F(T)$  (actually, a category of maps) from any test scheme  $T = \text{Spec}(A)$  to  $Uhl_n$ . Firstly, that would allow to define  $Uhl_n$  over any field  $k$  and not to require stability. Secondly, in the study of the cohomology of  $Uhl_n$  and the action of Hecke correspondences on it, one needs to deal with the phenomenon of unexpected dimension of  $Uhl_n$ . A possible approach involves defining a “derived moduli space”  $DUhl_n$  in the sense of [Lu] which would amount to considering more general “spaces”  $T$ . Thus, defining  $Uhl_n$  as a functor is a necessary preliminary step to constructing  $DUhl_n$ .

Very roughly, it is expected that a map  $T \rightarrow Uhl_n$  should be described by a vector bundle  $F$  on an open subset  $U \subset T \times S$  such that its complement  $Z$  is finite over  $T$ , a family  $\xi$  of effective zero cycles on  $X$  parametrized by  $T$  plus an agreement condition between  $\xi$  and  $F$ . Such a definition gives a “reasonable space”  $Uhl_n$  if it satisfies a criterion due to Artin, cf. [Ar], or its “derived” generalization proved in [Lu]. The most difficult part of Artin’s criterion is the effectiveness condition: if  $A$  is a complete noetherian local  $k$ -algebra with maximal ideal  $\mathfrak{m}$  and  $A_p = A/\mathfrak{m}^{p+1}$  one needs to show that  $F(\text{Spec}(A)) = \varprojlim F(\text{Spec}(A_p))$ . Ignoring the family of zero cycles  $\xi$  (as will be done in this paper), if  $X = \text{Spec}(A) \times_k S$  and  $\widehat{X}$  is its formal completion along the fiber over the closed point of  $\text{Spec}(A)$ , we are trying to find whether a bundle  $\mathcal{F}$  on an open subset  $\widehat{U} \subset \widehat{X}$  comes from a bundle  $F$  on an open subset  $U \subset X$ . Such  $F$  is called an *algebraization* of  $\mathcal{F}$ .

In this paper we prove that, when  $S$  has arbitrary dimension and  $\widehat{U}$  has complement of codimension  $\geq 3$ , algebraization always exists (for vector bundles and also for principal bundles over reductive groups). If  $\widehat{U}$  has complement of codimension  $\geq 2$  then algebraization exists only under an additional condition (which, in the Uhlenbeck functor case, is guaranteed due to the presence of the relative zero cycle  $\xi$ ).

Earlier similar questions were studied for coherent sheaves on proper schemes by Grothendieck, see [EGAIII], and in the case of Lefschetz type theorems by Grothendieck and Raynaud in [SGA2],

and [R]. Although these results do not apply in our case directly, our proof is based on the tools developed in [EGAIII], [SGA2].

In Section 2 we fix the notation, give examples illustrating some problems to be encountered, and prove algebraization results for vector bundles, summarized in Corollary 8. In Section 3 we formulate an algebraization criterion for principal bundles over reductive groups, see Theorem 9. Finally, Section 4 provides a categorical restatement of our results, see Theorem 13.

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## 2 Algebraization for vector bundles.

### 2.1 Setup

We refer the reader to Expose III in [SGA2] regarding basic properties of depth and its relation to local cohomology. Let  $S$  be an irreducible noetherian scheme of finite type over a field  $k$ . We will assume that  $S$  is proper and satisfies Serre's  $S_2$  condition: for any  $s \in S$ ,  $\text{depth}_s \mathcal{O}_S \geq \min(\dim \mathcal{O}_{S,s}, 2)$ . Let  $V \subset S$  be an open subset with closed complement of codimension  $\geq 2$  in  $S$  and  $A$  a complete noetherian local  $k$ -algebra with residue field  $K = A/\mathfrak{m}$  and associated graded  $K$ -algebra  $gr(A) = \bigoplus_{p \geq 0} gr_p(A) = \bigoplus_{p \geq 0} \mathfrak{m}^p / \mathfrak{m}^{p+1}$ . Define  $X = S \times_k \text{Spec}(A)$  and

$$X_p = S \times_k \text{Spec}(A/\mathfrak{m}^{p+1}); \quad U_p = V \times_k \text{Spec}(A/\mathfrak{m}^{p+1}). \quad p \geq 0$$

Let  $i_p : U_p \rightarrow X_p$  be the natural open embeddings. Denote by  $\widehat{X}$  the completion of  $X$  along  $X_0$ , which may be viewed at the limit of  $\{X_p\}_{p \geq 0}$ , cf. Section 10.6 in [EGAII]. The open subset  $U_0 \subset X_0$  defines an open formal subscheme  $\widehat{i} : \widehat{U} \rightarrow \widehat{X}$ , given by the limit of  $\{U_p\}_{p \geq 0}$ . The ideal sheaf of  $X_0$  in  $X$  will be denoted by  $\mathcal{J}_X$  and the closed subset  $X_0 \setminus U_0$  by  $Z_0$ . Finally,  $f : X \rightarrow \text{Spec}(A)$  is the natural proper projection and, for any  $s \in \text{Spec}(A)$ ,  $X_s$  stands for the fiber  $f^{-1}(s)$ .

Observe that  $X$  may no longer satisfy the  $S_2$  condition (since we made no depth assumptions on  $A$ ). However, for  $f(x) = s$  we can lift a regular sequence from  $\mathcal{O}_{X_s, x}$  to  $\mathcal{O}_{X, x}$  which gives

**Lemma 1** *For any  $x \in X$  with  $f(x) = s$ ,  $\text{depth } \mathcal{O}_{X, x} \geq \min(\dim \mathcal{O}_{X_s, x}, 2)$ .  $\square$*

Consider a vector bundle  $\mathcal{F}$  on  $\widehat{U}$ , i.e. a sequence of vector bundles  $F_p$  on  $U_p$  with isomorphisms

$$F_p|_{U_{p-1}} \simeq F_{p-1}; \quad p \geq 1. \quad (1)$$

**Definition.** We will say that a vector bundle  $\mathcal{F}$  on  $\widehat{U}$  *admits an algebraization* if there exists an open subset  $U \subset X$  with  $U \cap X_0 = U_0$  and a vector bundle  $F$  on  $U$  such that  $\mathcal{F}$  is isomorphic to the completion of  $F$ , i.e. for  $\mathcal{J}_U = \mathcal{J}_X|_U$  there exist isomorphisms  $F_p \simeq F/\mathcal{J}_U^{p+1}F$  compatible with (1). In Section 3 we apply similar terminology to principal bundles.

Let  $Z$  be the closed subset  $X \setminus U$  and  $i : U \hookrightarrow X$  the open embedding.

**Lemma 2** *Let  $\text{codim}_{X_0} Z_0 \geq 2$  and suppose that  $U \subset X$  is an open subset such that  $U \cap X_0 = U_0$ . For any  $s \in \text{Spec}(A)$  define  $Z_s = Z \cap X_s$ . Then  $\text{codim}_{X_s} Z_s \geq 2$  for all  $s \in \text{Spec} A$  and  $\text{codim}_X Z \geq 2$ .*

*Proof.* Since  $f$  is proper, the image  $f(\overline{Z}_s)$  contains the unique closed point  $s_0 \in \text{Spec}(A)$ . Therefore  $\overline{Z}_s \cap X_0 \subset Z_0$  is not empty. By semicontinuity of dimensions in the fibers we have  $\text{codim}_{X_s} Z_s \geq \text{codim}_{X_0}(\overline{Z}_s \cap X_0) \geq \text{codim}_{X_0} Z_0 = 2$ . The second assertion of the lemma follows from the first.  $\square$

In our discussion, we repeatedly use the following results

**Proposition 3** *In the notation introduced above*

- (i). *Completion along  $X_0$  induces an equivalence between the category of coherent sheaves on  $X$  and the category of coherent sheaves on the formal scheme  $\widehat{X}$ .*
- (ii). *For any locally free sheaf  $F$  (resp.  $F_0$ ) on  $U$  (resp.  $U_0$ ) its direct image  $i_* F$  (resp.  $(i_0)_* F_0$ ) is coherent. If  $\text{codim}_{X_0} Z_0 \geq 3$  then  $R^1(i_0)_* F_0$  is also coherent.*
- (iii). *Let  $E$  be a coherent sheaf on  $X$  and  $\psi : E \rightarrow i_* i^* E$  the canonical morphism. Then  $\psi$  is an isomorphism if and only if  $\text{depth}_x E \geq 2$  for any point  $x \in Z = X \setminus U$ .*

*Proof.* Part (i) follows from Corollary 5.1.6 in [EGAIII]. To check the coherence of  $i_* F$ , by Corollary VIII.2.3 in [SGA2] it suffices to check that  $\text{depth}_x F \geq 1$  for any point  $x \in U$  such that  $\overline{\{x\}} \cap Z$  has codimension 1 in  $\overline{\{x\}}$ . But Lemma 1 and local freeness of  $F$  imply that any  $x$  with  $\text{depth}_x F = 0$  must be generic in its fiber, and the Lemma 2 implies that  $\overline{\{x\}} \cap Z$  would in fact have codimension 2 in  $\overline{\{x\}}$ . The same proof applies to  $(i_0)_* F_0$ . If  $\text{codim}_{X_0} Z_0 \geq 3$  then the above argument can also be applied to  $R^1(i_0)_* F_0$  once we show that  $\text{depth}_x F_0 \geq 2$  for any  $x \in U_0$  such that  $\overline{\{x\}} \cap Z_0$  has codimension 1 in  $\overline{\{x\}}$ . But by  $S_2$  condition  $\text{depth}_x F_0 \leq 1$  can only hold for points  $x$  of codimension  $\leq 1$  in  $U_0$ , which would imply that  $\overline{\{x\}} \cap Z_0$  has codimension  $\geq 2$  in  $\overline{\{x\}}$ . This proves (ii). Part (iii) is a particular case of Corollary II.3.5 in *loc.cit.*  $\square$

## 2.2 Examples.

The first example with  $\text{codim}_{X_0} Z_0 = 3$  and  $K = k$  shows that one may not be able to take  $U = U_0 \times_k \text{Spec}(A)$ .

**Example 1.** Take  $S = X_0 = \mathbb{P}^3$  with homogeneous coordinates  $[x : y : z : w]$  and set  $V = U_0 = S \setminus [0 : 0 : 0 : 1]$ ,  $A = k[[t]]$  (formal power series in  $t$ ). Define vector bundles  $F_p$  as kernels of

$$\varphi_p : \mathcal{O}_{U_p}^{\oplus 3} \rightarrow \mathcal{O}(1)_{U_p}; \quad (s_1 \oplus s_2 \oplus s_3) \mapsto s_1 x + s_2 y + s_3(z - tw).$$

Observe that  $\varphi_p$  is surjective since  $t$  is nilpotent on  $U_p$  and  $[0 : 0 : 0 : 1] \notin U_p$ .

**Lemma 4** *The bundle  $\mathcal{F}$  admits no algebraization  $(U, F)$  with  $U = U_0 \times_k \text{Spec}(A)$ .*

*Proof.* Set  $F$  to be the kernel of morphism  $\varphi : \mathcal{O}^{\oplus 3} \rightarrow \mathcal{O}(1)$  of vector bundles on  $U$ , given by the same formula as for  $\varphi_p$ . By definition,  $\varphi$  is not surjective only at  $P = [0 : 0 : t : 1] \in U$  which projects to the generic point  $\xi = \text{Spec}(k[t^{-1}, t]) \in \text{Spec}(A)$ . The specialization at  $t = 0$  is not in  $U_0$ , hence  $P$  is closed in  $U$  and  $U \setminus P$  is an open subset containing  $U_0$ . Since on  $U \setminus P$  we have the short exact sequence of locally free sheaves

$$0 \rightarrow F \rightarrow \mathcal{O}^{\oplus 3} \rightarrow \mathcal{O}(1) \rightarrow 0,$$

the restriction of  $F$  to each  $U_p$  is given by  $F_p$ , i.e.  $\mathcal{F}$  is indeed the completion of  $F$ . On the other hand,  $F$  is not locally free at  $P$ : from  $0 \rightarrow F \rightarrow \mathcal{O}_U^{\oplus 3} \rightarrow \mathcal{O}_U \rightarrow k_P \rightarrow 0$  we immediately get  $\mathcal{E}xt^1(F, \mathcal{O}_U) \simeq \mathcal{E}xt^3(k_P, \mathcal{O}) \simeq k_P$  since the middle two terms are projective.

Suppose that  $E$  is a locally free sheaf on  $U$  with completion isomorphic to  $\mathcal{F}$ . We will see later in Proposition 7(ii) that in such situation we must have:  $\widehat{i_*E} \simeq \widehat{i_*\mathcal{F}} \simeq \widehat{i_*F}$  hence by Proposition 3(i),  $i_*F \simeq i_*E$  which contradicts  $\mathcal{E}xt^1(F, \mathcal{O}_U) \neq 0$ .  $\square$

The second example illustrates that for  $\text{codim}_{X_0} Z_0 = 2$ , a pair  $(U, F)$  may not exist at all.

**Example 2.** Consider  $A = k[[t]]$  and  $S = X_0 = \mathbb{P}^2$  with homogeneous coordinates  $(x : y : z)$ . Let  $V = U_0 = X_0 \setminus P$  where  $P = (0 : 0 : 1)$  and define a rank 2 bundle  $F_p$  on  $U_p = U_0 \times_k \text{Spec}(k[t]/t^{p+1})$  as follows. The affine open subsets  $U_p^{(x)}, U_p^{(y)}$  given by non-vanishing of  $x$ , resp.  $y$ , form a covering of  $U_p$  and we can glue trivial rank 2 bundles on these open sets, using the transition function

$$\begin{pmatrix} 1 & \sum_{m=0}^p \left(\frac{tz^2}{xy}\right)^m \\ 0 & 1 \end{pmatrix}$$

on  $U_p^{(x)} \cap U_p^{(y)}$ . Clearly  $F_p|_{U_{p-1}} \simeq F_{p-1}$  in a natural way, and we obtain a vector bundle  $\mathcal{F}$  on  $\widehat{U}$ .

**Lemma 5** *There exists no vector bundle  $F$  on  $U = X \setminus Z$  with  $\widehat{F} \simeq \mathcal{F}|_{\widehat{U} \setminus (Z \cap U_0)}$ , for any closed subset  $Z \subset X$  such that  $Z_0 \subset (Z \cap X_0)$  and  $\text{codim}_{X_0}(Z \cap X_0) \geq 2$ .*

*Proof.* Suppose otherwise and take the direct image of  $F$  with respect to the open embedding  $i : U \rightarrow X$ . By Proposition 3,  $i_*F$  is coherent and has *depth*  $\geq 2$  at all codimension 2 points of  $X$ . Since modules of depth 2 over two-dimensional regular local rings are free by Auslander-Buchsbaum formula,  $i_*F$  will be locally free in codimension two. Therefore shrinking  $Z$  we can assume that  $Z$  has codimension 3 in  $X$  which in our case means that  $Z$  is a finite set of points in  $X_0$ . Then the short exact sequence of sheaves on  $X \setminus Z$

$$0 \rightarrow F \xrightarrow{t^{p+1}} F \rightarrow F_p \rightarrow 0,$$

leads to a long exact sequence on  $X$ :

$$0 \rightarrow i_*F \xrightarrow{t^{p+1}} i_*F \rightarrow (i_p)_*F_p \rightarrow R^1i_*F \xrightarrow{t^{p+1}} R^1i_*F$$

where  $R^1i_*F$  is coherent for the same reason as in Proposition 3(ii). Since  $R^1i_*F$  is supported at the finite set  $Z$  of closed points, it has finite length at each of them and the last arrow is zero for  $p \geq p_0$ . For such  $p$  we can write  $i_*F \rightarrow (i_p)_*F_p \rightarrow R^1i_*F \rightarrow 0$  which gives

$$i_*F \otimes_{\mathcal{O}_X} k(P) \rightarrow (i_p)_*F_p \otimes_{\mathcal{O}_X} k(P) \rightarrow R^1i_*F \otimes_{\mathcal{O}_X} k(P) \rightarrow 0$$

To prove the lemma it suffices to show that  $\dim_k(i_p)_*F_p \otimes_{\mathcal{O}_X} k(P)$  is unbounded as  $p \rightarrow \infty$ .

To that end, replace  $X_0$  with the affine open subset  $\widetilde{X}_0 \simeq \mathbb{A}^2$  given by non-vanishing of  $z$ , with affine coordinates  $u = \frac{x}{z}, v = \frac{y}{z}$ . Set  $W_0 = U_0 \cap \widetilde{X}_0$  and similarly for  $\widetilde{X}_p, W_p, W_p^{(x)}$  and  $W_p^{(y)}$ . Then  $(i_p)_*F_p|_{\widetilde{X}_p}$  is the sheaf associated to  $H^0(W_p, F_p|_{W_p})$  viewed as a module over  $A(\widetilde{X}_p) = k[u, v, t]/t^{p+1}$ . By its definition,  $F_p$  is an extension of  $\mathcal{O}_{U_p}$  with  $\mathcal{O}_{U_p}$  which leads to long exact sequence

$$0 \rightarrow H^0(W_p, \mathcal{O}_{W_p}) \rightarrow H^0(W_p, F_p|_{W_p}) \rightarrow H^0(W_p, \mathcal{O}_{W_p}) \rightarrow H^1(W_p, \mathcal{O}_{W_p}).$$

where the last arrow sends the constant function 1 to the class of the extension. Let  $M_p$  be the kernel of the last arrow. It suffices to show that  $\dim_k(M_p/\langle u, v, t \rangle M_p)$  is unbounded. Computing  $M_p$  via the affine covering  $\{W_p^{(x)}, W_p^{(y)}\}$  we identify it with the kernel of

$$k[u, v, t]/t^{p+1} \xrightarrow{\pi_p \circ \psi_p} \frac{1}{uv} k[u^{-1}, v^{-1}, t]/t^{p+1}$$

where  $\psi_p$  is multiplication by  $\sum_{l=0}^p (\frac{t}{uv})^l$  (i.e. the upper right corner of the transition matrix in the definition of  $F_p$ ), and  $\pi_p$  is the natural projection

$$k[u, u^{-1}, v, v^{-1}, t]/t^{p+1} \rightarrow \frac{1}{uv} k[u^{-1}, v^{-1}, t]/t^{p+1}$$

It follows that  $M_p$  is generated by the monomials  $t^p, t^i u^{p-i}, t^i v^{p-i}$  for  $i = 0, \dots, p-1$ , thus

$$\dim_k(M_p/\langle u, v, t \rangle M_p) = 2p + 1 \rightarrow \infty \quad \text{as } p \rightarrow \infty. \quad \square$$

**Example 3.** (Suggested to the author by V. Drinfeld.) The bundle in the previous example has trivial determinant, but if we don't insist on this condition, then there is a rank one example: glue two trivial bundles on  $U_p^{(x)}, U_p^{(y)}$  using the transition function  $\sum_{m=0}^p (\frac{tz^2}{xy})^m$ . The resulting line bundle admits no algebraization since again  $\dim_k(i_p)_* F_p \otimes_{\mathcal{O}_X} k(P)$  is not bounded as  $p \rightarrow \infty$ .

## 2.3 Algebraization of vector bundles.

**Theorem 6** *In the notation of section 2.1,*

- (i). *If  $\text{codim}_{X_0} Z_0 \geq 3$  then  $\mathcal{F}$  admits an algebraization.*
- (ii). *If  $\text{codim}_{X_0} Z_0 \geq 2$  and the cokernel of the natural morphism  $(i_p)_* F_p|_{X_{p-1}} \rightarrow (i_{p-1})_* F_{p-1}$  is supported in codimension  $\geq 3$  for all  $p$  large enough, then  $\mathcal{F}$  admits an algebraization.*
- (iii). *In either of the two situations (codimension  $\geq 3$  or codimension  $\geq 2$  with the additional support assumption) the projective system  $\{(i_p)_* F_p\}_{p \geq 0}$  satisfies the Mittag-Leffler condition, the direct image  $\widehat{i}_* \mathcal{F}$  is coherent and isomorphic to  $\varprojlim (i_p)_* F_p$ .*

*Proof.* We split the proof of (i) and (ii) in a number of steps. Part (iii) will follow from Step 2.

*Step 1.*

Suppose that  $\widehat{i}_* \mathcal{F}$  is coherent. By Proposition 3(i) there exists a unique coherent sheaf  $E$  on  $X$  such that  $\widehat{E} \simeq \widehat{i}_* \mathcal{F}$ . The subset  $U \subset X$  of points where  $E$  is locally free is open and contains  $U_0$  (e.g. by Nakayama's Lemma). Shrinking  $U$  if necessary we can achieve  $U \cap X_0 = U_0$ . Now set  $F = E|_U$ .

*Step 2.*

Therefore (i) and (ii) are reduced to showing that, under the conditions stated,  $\widehat{i}_* \mathcal{F}$  is coherent. To that end we modify the argument of 0.13.7.7 in [EGAIII] which will also prove (iii). First, as in 0.13.7.2 of *loc. cit.*, we choose injective resolutions  $F_k \rightarrow L_k^\bullet$  such that  $L_{k+1}^\bullet / \mathcal{J}_U^{k+1} L_{k+1}^\bullet \simeq L_k^\bullet$  and the natural filtrations by  $\mathcal{J}_U^n(\dots)$  agree with those on  $F_k$ . Each  $\widehat{i}_*(L_k^\bullet)$  is a filtered complex and has a spectral sequence with  $E_1$  term given by

$$E_1^{pq} = R^{p+q} \widehat{i}_*(\mathcal{J}_U^p F_k / \mathcal{J}_U^{p+1} F_k)$$

As in 0.13.7.3 of *loc.cit.* we pass to the limit as  $k \rightarrow \infty$  and get a spectral sequence with

$$E_1^{p,q} = R^{p+q}\widehat{i}_*(F_p/F_{p+1}) \simeq R^{p+q}\widehat{i}_*(F_0) \otimes_K (\mathfrak{m}^p/\mathfrak{m}^{p+1}) = R^{p+q}\widehat{i}_*(F_0) \otimes_K gr_p(A)$$

We are interested in the components

$$E_1^0 = \bigoplus_{p+q=0} E_1^{p,q} = \widehat{i}_*(F_0) \otimes_K gr(A); \quad E_1^1 = \bigoplus_{p+q=1} E_1^{p,q} = R^1\widehat{i}_*(F_0) \otimes_K gr(A).$$

We would like to show that the spectral sequence converges at the  $E^0 = \bigoplus E^{p,-p}$  terms. Note that each  $E_{k+1}^1 = \bigoplus E_{k+1}^{p,1-p}$  is a quotient of a subsheaf in  $E_k^1$  while each  $E_{k+1}^0$  is a subsheaf  $E_k^0$  (since  $E^{p,-1-p}$  terms are zero). Taking successive preimages of the boundaries in  $E_{r-1}, E_{r-2}, \dots, E_1$  we get a sequence of boundary subsheaves  $B_1 \subset B_2 \subset B_3 \subset \dots \subset E_1^1$ , and taking preimages of cycles in  $E_k$  we get a sequence of cycle subsheaves  $E_1^0 \supset Z_1 \supset Z_2 \supset Z_3 \supset \dots$ . By 0.13.7.6 in *loc.cit.* these are actually  $\mathcal{O}_{X_0} \otimes_K gr(A)$ -submodules.

Suppose that sequence of cycles stabilizes, i.e. for some  $r_0$  one has  $Z_r = Z_{r_0}$  whenever  $r \geq r_0$ , then by 0.13.7.4 in [EGAIII], the projective system  $\{\widehat{i}_*(F_k)\}_{k \geq 0}$  satisfies the Mittag-Leffler condition and the associated graded of  $\widehat{i}_*(\mathcal{F})$  is precisely  $Z_{r_0} \subset \widehat{i}_*(F_0) \otimes_K gr(A)$ . But  $\widehat{i}_*(F_0)$  is a coherent by Proposition 3(ii), hence the subsheaf  $gr(\widehat{i}_*\mathcal{F}) \subset \widehat{i}_*(F_0) \otimes_K gr(A)$  is a coherent  $\mathcal{O}_{X_0} \otimes_K gr(A)$ -module, by the noetherian property of  $X_0$  and  $A$ . By *loc.cit.* 13.7.7.2,  $\widehat{i}_*\mathcal{F}$  is itself coherent on  $\widehat{X}$ . Also,  $\widehat{i}_*\mathcal{F} \simeq \varprojlim (i_p)_*F_p$  by 0.13.7.5.1 in *loc.cit.*

*Step 3.*

Now the assertion of the theorem is reduced to showing that the sequence of cycles  $Z_1 \supset Z_2 \supset \dots$  stabilizes. By definition of  $Z_i$  this is equivalent to saying that the higher differentials of the spectral sequence  $d_r : E_r^0 \rightarrow E_r^1$  become zero for  $r \geq r_0$ . That in turn is equivalent to saying that the sequence of boundaries  $B_1 \subset B_2 \subset B_3 \subset \dots$ , also stabilizes.

If  $\text{codim}_{X_0} Z_0 \geq 3$  by Proposition 3(ii),  $R^1(i_0)_*F_0$  is also coherent and  $\{B_r\}_{r \geq 1}$  stabilizes by the noetherian property of  $R^1(i_0)_*F_0 \otimes_K gr(A)$ , which proves (i). If  $\text{codim}_{X_0} Z_0 \geq 2$  we need to find a coherent subsheaf of  $R^1(i_0)_*F_0 \otimes_K gr(A)$  containing  $B_r$  for all  $r \geq 1$ .

*Step 4.*

At this point we reduced (ii) to showing that, under the assumptions stated, there exists a coherent subsheaf  $G \subset R^1(i_0)_*F_0$  such that  $B_r \subset G \otimes_K gr(A)$  for all  $r$ . By 0.11.2.2 in [EGAIII] for  $r \geq p$  the term  $B_r^{p,1-p}$  is the image of the connecting homomorphism

$$\widehat{i}_*F_p \rightarrow \widehat{i}_*F_{p-1} \xrightarrow{\rho_p} R^1\widehat{i}_*F_0 \otimes_K (\mathfrak{m}^p/\mathfrak{m}^{p+1})$$

in the long exact sequence obtained by applying  $R\widehat{i}_*$  to the short exact sequence on  $\widehat{U}$ :

$$0 \rightarrow F_0 \otimes_K (\mathfrak{m}^p/\mathfrak{m}^{p+1}) \rightarrow F_p \rightarrow F_{p-1} \rightarrow 0.$$

Observe that by our assumptions each  $\text{Im}(\rho_p)$  is coherent, and supported in codimension  $\geq 3$  for  $p \gg 0$ . Therefore we are done once we show that the subsheaf of  $R^1(i_0)_*F_0$  formed by all sections with support in codimension  $\geq 3$ , is coherent whenever  $\text{codim}_{X_0} Z_0 \geq 2$  and  $F_0$  is locally free on  $U_0$ .

*Step 5.*

Set  $Q = (i_0)_*F_0$ , a coherent sheaf on  $X_0$  by Step 2. By the standard exact sequence we have  $\mathcal{H}_{Z_0}^2 Q = R^1(i_0)_*Q|_{U_0} = R^1(i_0)_*F_0$ , so it suffices to show that  $\mathcal{H}_{\geq 3}^0 \mathcal{H}_{Z_0}^2 Q$  is coherent where  $\mathcal{H}_{\geq 3}^0$  is the functor of sections supported in codimension  $\geq 3$ . Let  $\mathcal{H}_{\geq 3}^i$  be the higher derived functors.

First, the standard spectral sequence for the composition of functors  $R\mathcal{H}_{\geq 3}^0, R\mathcal{H}_{Z_0}^0$  has  $E_2^{p,q} = \mathcal{H}_{\geq 3}^p \mathcal{H}_{Z_0}^q Q$ . But  $\mathcal{H}_{Z_0}^i Q = 0$  for  $i = 0, 1$  by Proposition 3(iii), so

$$\mathcal{H}_{\geq 3}^0 \mathcal{H}_{Z_0}^2 Q \simeq \mathcal{H}_{\Phi}^2 Q$$

where  $\mathcal{H}_{\Phi}^2$  is the local cohomology with the family of supports  $\Phi$  formed by all codimension  $\geq 3$  closed subsets in  $Z_0$ .

*Step 6.*

To show that  $\mathcal{H}_{\Phi}^2 Q$  is coherent let  $\omega$  be the dualizing complex of  $X_0$ , cf. [Ha2]. More precisely, by *loc.cit*  $\omega$  is quasi-isomorphic to a complex of injective sheaves

$$0 \rightarrow \mathcal{K}^0 \rightarrow \dots \rightarrow \mathcal{K}^{\dim_K X_0} \rightarrow 0$$

with  $\mathcal{K}^i = \bigoplus_{\text{codim}_{X_0} x=i} i_*^x(I(x))$  and  $i^x : \text{Spec}(\mathcal{O}_{X_0,x}) \rightarrow X_0$  is the natural morphism, while  $I(x)$  is an injective envelope of the residue field  $k(x)$  as a module over  $\mathcal{O}_{X_0,x}$ . An easy but important observation which we use below, is that  $\mathcal{K}^p$  has no sections supported in codimension  $\geq p+1$ .

By definition of a dualizing complex, the double complex  $\mathcal{K}^{p,q} = \mathcal{H}om(\mathcal{H}om(Q, \mathcal{K}^q), \mathcal{K}^p)$  has total complex quasi-isomorphic to  $Q$ . Moreover, by Proposition IV.2.1 and the remark on page 123 in [Ha2], this total complex is a flasque resolution of  $Q$  and hence can be used to compute  $\mathcal{H}_{\Phi}^{\bullet}(Q)$ . This leads to a spectral sequence:

$$E_2^{p,q} = \mathcal{E}xt_{\Phi}^p(\mathcal{E}xt^{-q}(Q, \omega), \omega) \Rightarrow \mathcal{H}_{\Phi}^{p+q}(Q)$$

where  $\mathcal{E}xt_{\Phi}^p = R^p(\Gamma_{\Phi} \circ \mathcal{H}om)$  and the  $\mathcal{E}xt$  sheaves are understood in the sense of hypercohomology.

Since  $E_2^{p,q} \neq 0$  only for when  $p$  and  $(-q)$  are between 0 and  $\dim_K X_0$  only finitely many terms with fixed  $p+q$  will be non-trivial and to show that  $\mathcal{H}_{\Phi}^2(Q)$  is coherent it suffices to show that  $\mathcal{E}xt_{\Phi}^p(\mathcal{E}xt^{p-2}(Q, \omega), \omega)$  is coherent for  $p \geq 2$ .

*Step 7.*

First observe that  $\mathcal{E}xt_{\Phi}^2(G, \omega) = 0$  for any quasi-coherent sheaf  $G$  since  $\mathcal{K}^2$  has no sections supported in codimension  $\geq 3$  and hence no sections with support in  $\Phi$ . Hence we can assume that  $p \geq 3$ .

Denote  $\mathcal{R}^p = \mathcal{E}xt^{p-2}(Q, \omega)$ . We first claim that  $\text{codim}_{X_0} \text{Supp}(\mathcal{R}^p) = d \geq p \geq 3$ . In fact, let  $x \in \text{Supp}(\mathcal{R}^p)$  be a point with  $\dim \mathcal{O}_{X_0,x} = d$ . Since the stalk of  $\mathcal{R}_x^p$  is non zero, and by local duality, cf. V.6 in [Ha2] its completion is dual to  $\mathcal{H}_x^{d+2-p}(Q)$  we conclude that  $\mathcal{H}_x^{d+2-p}(Q) \neq 0$ . Then  $d+2-p \geq 0$  and  $d \geq p-2 \geq 1$ . If  $d=1$  then  $p=3$  and also  $x \notin Z_0$  hence the stalk  $Q_x$  is free. Thus  $\mathcal{H}_x^0(\mathcal{O}) \neq 0$ , contradicting the  $S_2$  assumption. If  $d \geq 2$  then applying the  $S_2$  condition when  $x \notin Z_0$  and Proposition 3(iii) when  $x \in Z_0$  we actually have  $d+2-p \geq 2$  so  $d \geq p$  as required.

By primary decomposition, the coherent sheaf  $\mathcal{R}^p$  admits a finite filtration by coherent subsheaves such that all successive quotients have irreducible supports of codimension  $\geq p$ . By the standard long exact sequence for  $\mathcal{E}xt_{\Phi}^{\bullet}(\cdot, \omega)$  it suffices to show that  $\mathcal{E}xt_{\Phi}^p(G, \omega)$  is coherent whenever  $p \geq 3$  and  $G$  is a coherent sheaf with irreducible support  $Y$  of codimension  $\geq p$ .

If  $Y \not\subseteq Z_0$  for any  $W$  in the family  $\Phi$ , the intersection  $Y \cap W$  is not equal to  $Y$  and therefore has codimension  $\geq p+1$ . But then  $\mathcal{E}xt_{\Phi}^p(G, \omega) = 0$  because any section  $\rho$  of  $\mathcal{H}om(G, \mathcal{K}^p)$  representing a class in  $\mathcal{E}xt_{\Phi}^p(G, \omega)$  has zero values since  $\mathcal{K}^p$  has no sections supported in codimension  $\geq p+1$ . If  $Y \subseteq Z_0$  then  $Y$  is an element of  $\Phi$  and  $\mathcal{E}xt_{\Phi}^p(G, \omega) \simeq \mathcal{E}xt^p(G, \omega)$  since all sections of  $\mathcal{H}om(G, \mathcal{K}^t)$  have support in  $\Phi$ . But  $\mathcal{E}xt^p(G, \omega)$  is coherent which finishes the proof.  $\square$

The converse to Theorem 6 can be formulated as follows.

**Proposition 7** *In the setting of Section 2.1, assume that  $\mathcal{F}$  admits an algebraization  $(U, F)$  and view each  $F_p$  as a sheaf on  $U$ . Then*

(i). *The cokernel of  $i_*F_p \rightarrow i_*F_{p-1}$  is supported in codimension  $\geq 3$  for  $p \gg 0$ .*

(ii). *The isomorphism  $\widehat{F} \simeq \mathcal{F}$  extends to direct images:  $\widehat{i_*F} \simeq \widehat{i_*\mathcal{F}}$ . In particular,  $\widehat{i_*\mathcal{F}}$  is coherent.*

*Proof.* To prove (i) observe that the cokernel of  $i_*F_p \rightarrow i_*F_{p-1}$  is annihilated by  $\mathcal{J}_X$ , being a subsheaf of  $R^1i_*F_0 \otimes_K gr_p(A)$ , and is therefore isomorphic to the cokernel of  $i_*F_p|_{X_0} \rightarrow i_*F_{p-1}|_{X_0}$ .

We will first show that the natural map  $i_*F_p|_{X_0} \rightarrow i_*F_0$  is an embedding of sheaves for all  $p$ . Considering the exact sequence

$$0 \rightarrow \mathcal{J}_X(i_*F_p) \rightarrow i_*F_p \rightarrow i_*F_p|_{X_0} \rightarrow 0$$

and its map to the first terms of the sequence

$$0 \rightarrow i_*(\mathcal{J}_UF_p) \rightarrow i_*F_p \rightarrow i_*F_0 \rightarrow R^1i_*(\mathcal{J}_UF_p) \rightarrow \dots$$

we see that  $i_*F_p|_{X_0} \rightarrow i_*F_0$  is an embedding precisely when the natural map  $\mathcal{J}_X(i_*F_p) \rightarrow i_*(\mathcal{J}_UF_p)$  is an isomorphism. Observe that  $i_*\mathcal{O}_U = \mathcal{O}_X$  hence  $i_*\mathcal{J}_U$  is a sheaf of ideals in  $\mathcal{O}_X$ .

Using Lemma 1 and the Cohen-Macaulay assumption on  $X_0$  we see that  $\mathcal{H}_Z^t\mathcal{O}_X = \mathcal{H}_{Z_0}^t\mathcal{O}_{X_0} = 0$  for  $t = 0, 1$ . By the short exact sequence  $0 \rightarrow \mathcal{J}_X \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_{X_0} \rightarrow 0$  we derive  $\mathcal{H}_Z^t\mathcal{J}_X = 0$  for  $t = 0, 1$  and hence  $\mathcal{J}_X = i_*\mathcal{J}_U$  by Proposition 3 (iii). Then

$$i_*(\mathcal{J}_UF_p) = (i_*\mathcal{J}_U)(i_*F_p) = \mathcal{J}_Xi_*F_p$$

as required. Similarly,  $i_*F|_{X_0} \rightarrow i_*F_0$  is an embedding. So for any  $p \geq 1$  we have embeddings

$$i_*F|_{X_0} \hookrightarrow i_*F_p|_{X_0} \hookrightarrow i_*F_{p-1}|_{X_0} \hookrightarrow i_*F_0$$

Consequently, the coherent sheaf  $\mathcal{K} = \text{Coker}(i_*(F)|_{X_0} \rightarrow i_*F_0)$  has a decreasing filtration by images of  $i_*F_p|_{X_0}$  and each  $\text{Coker}(i_*F_p|_{X_0} \rightarrow i_*F_{p-1}|_{X_0})$  is its successive quotient. But  $\mathcal{K}$  is a coherent sheaf with  $\text{Supp}(\mathcal{K}) \subset Z_0$  and  $Z_0$  has at most finitely many points of codimension 2. Since for each point  $x \in X_0$  of codimension 2, the localization  $\mathcal{K}_x$  is a module of finite length, only finitely many successive quotients of the filtration of  $\mathcal{K}$  can be non-trivial in codimension 2, which proves (i).

To prove (ii) first observe that  $\widehat{i_*\mathcal{F}}$  and  $E = i_*F$  are coherent by Theorem 6(iii) and Proposition 3(ii), respectively. By Proposition 3(i) we can find a sheaf  $E'$  such that  $\widehat{E'} \simeq \widehat{i_*\mathcal{F}}$ . The isomorphism  $\widehat{E}|_{\widehat{U}} \simeq \mathcal{F} = \widehat{i_*\mathcal{F}}$  extends uniquely to a morphism of sheaves  $\widehat{\phi} : \widehat{E} \rightarrow \widehat{i_*\mathcal{F}} = \widehat{E'}$ . By Proposition 3(i),  $\widehat{\phi}$  is the completion of a unique morphism  $\phi : E \rightarrow E'$  which by Corollary 10.8.14 in [EGA1] should be an isomorphism on an open subset  $W$  containing  $U_0$ . Shrinking  $W$  if necessary we can assume  $W \subset U$ . By Lemma 2, each point  $x \in U \setminus W$  has codimension  $\geq 2$  in its fiber, hence  $\text{depth}_xE \geq 2$  by Lemma 1. For  $x \in X \setminus U$  we still have  $\text{depth}_xE \geq 2$  by Proposition 3(iii). Applying the same result to  $j : W \hookrightarrow X$  instead of  $U$  we see that  $E = j_*j^*E$ . By adjunction of  $j^*$  and  $j_*$  the isomorphism  $(\phi|_W)^{-1} : j^*E' \rightarrow j^*E$  extends uniquely to a morphism  $\psi : E' \rightarrow j_*j^*E = E$ .

By construction, the composition  $\psi\phi : E \rightarrow E$  restricts to identity on  $W$  hence  $\psi\phi = \text{Id}_E$ , by the same adjunction. Similarly, the composition  $\widehat{\phi}\widehat{\psi} : \widehat{E} \rightarrow \widehat{E'}$  restricts to identity on  $\widehat{U}$  and since  $\widehat{E'} \simeq \widehat{i_*\mathcal{F}}$ , we must have  $\widehat{\phi}\widehat{\psi} = \text{Id}_{\widehat{E'}}$ , so  $\phi\psi = \text{Id}_{E'}$  by Proposition 3(i). We have proved that  $E = i_*F \simeq E'$ . Since  $\widehat{E'} = \widehat{i_*\mathcal{F}}$  we conclude that  $\widehat{i_*F} = \widehat{i_*\mathcal{F}}$ .  $\square$



**Corollary 8** *The following conditions are equivalent:*

- (i). *The cokernel of  $(i_p)_*F_p \rightarrow (i_{p-1})_*F_{p-1}$  is supported in codimension  $\geq 3$  for  $p \gg 0$ .*
- (ii). *The projective system  $\{\hat{i}_*F_p\}_{p \geq 1}$  satisfies the Mittag-Leffler condition.*
- (iii). *The direct image  $\hat{i}_*\mathcal{F}$  is coherent.*
- (iv). *The bundle  $\mathcal{F}$  admits an algebraization.*

*Proof.* The implications (i)  $\Rightarrow$  (ii) and (iii)  $\Rightarrow$  (iv) are established in the proof of Theorem 6. The implication (iv)  $\Rightarrow$  (i) is proved in Proposition 7. If the projective system  $\{\hat{i}_*F_p\}_{p \geq 1}$  satisfies the Mittag-Leffler condition, by 0.13.3.1 in [EGAIII] the natural map  $\hat{i}_*\mathcal{F} \rightarrow \varprojlim \hat{i}_*F_p$  is an isomorphism. By the Mittag-Leffler condition we can replace  $\hat{i}_*F_p$  by a system of subsheaves  $G_p \subset \hat{i}_*F_p$  so that the property  $\hat{i}_*\mathcal{F} \simeq \varprojlim G_p$  still holds and  $G_p|_{X_{p-1}} \rightarrow G_{p-1}$  is surjective. Since each  $G_p$  is coherent by the noetherian property of  $X_p$ , Proposition 10.11.3 in [EGAII] tells that  $\varprojlim G_p$  is also coherent. Therefore, (ii)  $\Rightarrow$  (iii).  $\square$

**Remark.** Suppose that  $X_0$  is a smooth projective surface over  $K$ ,  $\xi = k_1P_1 + \dots + k_lP_l$  an effective zero cycle and  $F_0$  a rank  $n$  vector bundle on  $U_0 = X_0 \setminus \{P_1, \dots, P_l\}$ . The pair  $(F_0, \xi_0)$  should define a point  $\text{Spec}(K) \rightarrow \text{Uhl}_n$  of the Uhlenbeck functor. Assume that  $(F, \xi) : \text{Spec}(A) \rightarrow \text{Uhl}_n$  extends  $(F_0, \xi_0)$ . Then it is expected that  $\text{Coker}(i_*F \rightarrow i_*F_0)$  can be supported only at the points  $P_1, \dots, P_l$ , with multiplicities bounded by  $k_1, \dots, k_l$ , respectively (in the differential geometry picture, cf. [DK],  $\xi_0$  represents the singular part of a connection which may be smoothed out by  $F$  but may not acquire any negative coefficients; since the multiplicities of  $\text{Coker}(i_*F \rightarrow i_*F_0)$  measure the local change of  $c_2$  one obtains the bound mentioned). But the proof of Proposition 7 shows that the multiplicities of  $\text{Coker}(i_*F \rightarrow i_*F_0)$  give an upper bound for the total sum, over all  $p$ , of similar multiplicities for  $\text{Coker}((i_p)_*F_p \rightarrow (i_{p-1})_*F_{p-1})$ . Hence the condition of Corollary 8(i) is rather natural from the point of view of Uhlenbeck spaces.

### 3 Algebraization of principal bundles.

Let  $G$  be an affine algebraic group over  $k$ . We keep the notation of Section 2.1. and consider left principal  $G$ -bundles which are locally trivial in fppf topology. For such a  $G$ -bundle  $P$  (over  $\widehat{U}$  or an open subset  $U \subset X$ ) and any scheme  $Y$  over  $k$  with left  $G$ -action, denote by  $P_Y = G \backslash (Y \times_k P)$  the associated fiber bundle, i.e. the quotient by the left diagonal action of  $G$ . For instance, when  $\rho : G \rightarrow H$  is a homomorphism of linear algebraic groups over  $k$ , we can consider a left  $G$ -action on  $H$  given by  $g \cdot h = h\rho(g)^{-1}$  and then  $P_H$  is simply the principal  $H$ -bundle induced via  $\rho$ .

**Theorem 9** *Assume that the identity component  $G^\circ$  is reductive. Then a principal  $G$ -bundle  $\mathcal{P}$  over the formal scheme  $\widehat{U}$  admits an algebraization if and only if for a fixed exact representation  $G \hookrightarrow GL(V)$  the associated vector bundle  $\mathcal{P}_V$  admits an algebraization, i.e. satisfies the conditions of Corollary 8.*

The “only if” part is obvious. Since by a result of Haboush, cf. Theorem 3.3 in [Ha1], the quotient  $GL(V)/\eta(G)$  is affine, the “if” part follows from the following general statement.

**Proposition 10** *Let  $H$  an affine algebraic group over  $k$  and  $G$  its closed subgroup such that  $H/G$  is affine. Suppose that  $\mathcal{P}$  is a principal  $G$ -bundle over  $\widehat{U}$  such that the associated principal  $H$ -bundle  $\mathcal{Q} = \mathcal{P}_H$  admits an algebraization. Then  $\mathcal{P}$  admits an algebraization.*

First we establish a preparatory result. As before,  $U \subset X$  is an open subset satisfying  $U \cap X_0 = U_0$ .

**Lemma 11** *Let  $H$  be a linear algebraic group over  $k$ ,  $Q$  be a principal  $H$ -bundle on  $U$  and  $\widehat{Q}$  its completion. Let also  $Y$  be an affine  $H$ -variety. Then for any section  $\widehat{s} : \widehat{U} \rightarrow \widehat{Q}_Y$  there exists a section  $s : W \rightarrow Q_Y$  on an open subset  $W \subset U$  containing  $U_0$ , with completion equal to  $\widehat{s}$ . If  $(W, s)$  and  $(W', s')$  are two such algebraizations, then  $s = s'$  on  $W \cap W'$ .*

*Proof.* One can find a  $H$ -invariant linear subspace  $V^\vee \subset k[Y]$  containing a set of generators of  $k[Y]$  as a  $k$ -algebra. Then the surjection  $\text{Sym}_k^*(V^\vee) \rightarrow k[Y]$  gives an  $H$ -equivariant closed embedding  $Y \rightarrow V$  into the dual space  $V$ . This induces closed embeddings  $Q_Y \rightarrow Q_V$  and  $\widehat{Q}_Y \rightarrow \widehat{Q}_V$ .

Therefore  $\widehat{s}$  becomes a section of the vector bundle  $\widehat{Q}_V$ . By Proposition 7(ii) the completion of the coherent sheaf  $i_*Q_V$  is isomorphic to  $\widehat{i}_*\widehat{Q}_V$  and therefore by Proposition 3(i) there exists a unique section  $\tilde{s}$  of  $i_*Q_V$  with completion given by  $\widehat{i}_*\widehat{s}$ . Set  $s = \tilde{s}|_U$ .

It remains to show that  $s(W) \subset Q_V$  on some  $W$  as above. Let  $\mathcal{A} = \text{Sym}^*(Q_V^\vee)$  be the sheaf of symmetric algebras on  $U$  corresponding to  $Q_V$  and  $\mathcal{I} \subset \mathcal{A}$  the ideal sheaf of  $Q_Y$ . The section  $s$  gives the evaluation morphism  $\rho : \mathcal{A} \rightarrow \mathcal{O}_U$ . The sheaf  $G = \rho(\mathcal{I})$  is coherent, being a subsheaf of  $\mathcal{O}_U$ . Since  $\widehat{s}$  takes values in  $\widehat{Q}_Y$ , the completion  $\widehat{G}$  is zero. By Corollary 10.8.12 in [EGAI] this implies  $\text{Supp}(G) \cap U_0 = \emptyset$  hence  $W = U \setminus \text{Supp}(G)$  satisfies the conditions of the lemma. The uniqueness of  $s$  follows from the uniqueness of  $\tilde{s}$ .  $\square$

*Proof of Proposition 10.* Let  $(U, Q)$  be an algebraization of  $\mathcal{Q}$ . In general, giving a principal  $G$ -bundle is equivalent to giving a principal  $H$ -bundle  $\mathcal{R}$  together with a reduction to  $G$ , i.e. a section of the associated bundle  $\mathcal{R}_{H/G}$  with the fiber  $H/G$ . Since  $\mathcal{Q}$  is induced from  $\mathcal{P}$ , we get a section  $\widehat{s} : \widehat{U} \rightarrow \mathcal{Q}_{H/G}$  and by the above lemma there exists  $s : W \rightarrow Q_{H/G}$  such that  $\widehat{s}$  is equal to its completion. Then  $\mathcal{P}$  admits an algebraization  $(W, P)$  where  $P$  is the pullback of the principal  $G$ -bundle  $Q \rightarrow Q_{H/G}$  via  $s : W \rightarrow Q_{H/G}$ .  $\square$

## 4 Categorical formulations.

**Proposition 12** *The functor  $F \mapsto \widehat{F}|_{\widehat{U}}$  induces an equivalence between the full subcategory of all coherent sheaves  $E$  on  $X$  which are locally free at the points of  $U_0 \subset X$  and have  $\text{depth}_x E \geq 2$  at the points where  $E$  is not locally free, and the full subcategory of locally free sheaves on  $\widehat{U}$  admitting algebraization.*

*Proof.* Let  $(U, F)$  be an algebraization of  $\mathcal{F}$ . Then the sheaf  $E = i_*F$  satisfies  $E \simeq i_*i^*E$  hence by Proposition 3(iii)  $\text{depth}_x E \geq 2$  for all  $x \in Z = X \setminus U$ . We also observe that  $E$  is uniquely determined by  $\mathcal{F}$ , since by Propositions 3(i) and 7(ii) it is the unique coherent sheaf on  $X$  such that  $\widehat{E} \simeq \widehat{i}_*\mathcal{F}$ . Thus the functor described is essentially surjective on objects. For the morphisms, let  $\mathcal{F}_1, \mathcal{F}_2$  be a pair of vector bundles on  $\widehat{U}$  with algebraizations  $(U, F_1)$  and  $(U, F_2)$ , respectively, which we may assume to be defined on the same  $U$ . Denote by  $E_1 = i_*F_1, E_2 = i_*F_2$  the corresponding coherent sheaves on  $X$ . Then  $\text{Hom}_{\widehat{U}}(\mathcal{F}_1, \mathcal{F}_2) = \text{Hom}_{\widehat{X}}(\widehat{i}_*\mathcal{F}_1, \widehat{i}_*\mathcal{F}_2) = \text{Hom}_X(E_1, E_2)$  where the first equality is by adjunction of  $i^*$  and  $i_*$  and the second by Propositions 3(i) and 7(ii).  $\square$

To formulate a result for principal bundles, let  $\mathcal{B}(G, U_0)$  be the groupoid category in which the objects are given by pairs  $(U, P)$  where  $U \subset X$  is an open subset with  $U \cap X_0 = U_0$ , and  $P$  is a principal  $G$ -bundle on  $U$ . Morphisms from  $(U, P)$  to  $(U', P')$  are given by the set of equivalence classes of pairs  $(W, \psi)$  where  $W \subset U \cap U'$  is an open subset with  $W \cap X_0 = U_0$  and  $\psi : P|_W \rightarrow P'|_W$  an isomorphism of  $G$ -bundles. Two such pairs  $(W, \psi)$  and  $(W, \psi')$  are equivalent if  $\psi = \psi'$  on  $W \cap W'$ . Also denote by  $Bun(G, \widehat{U})$  the groupoid category of  $G$ -bundles on the formal scheme  $\widehat{U}$ . Completion along  $U_0$  defines a functor  $\Psi : \mathcal{B}(G, U_0) \rightarrow Bun(G, \widehat{U})$ . The following statement summarizes our results on algebraization of principal bundles

**Theorem 13** *With the notation of Section 2.1,*

- (i). *For any affine algebraic group  $G$  over  $k$ ,  $\Psi : \mathcal{B}(G, U_0) \rightarrow Bun(G, \widehat{U})$  is full and strict.*
- (ii). *For  $G = GL_n(k)$  the essential image of  $\Psi$  is the full subcategory of rank  $n$  vector bundles  $\mathcal{F} = \varprojlim F_p$  on  $\widehat{U}$  which satisfy the equivalent conditions (i)-(iii) of Corollary 8.*
- (iii). *Let  $G \hookrightarrow H$  be a closed embedding of affine algebraic groups over  $k$  such that  $H/G$  is affine. Then the natural functor from  $G$ -bundles to  $H$ -bundles induces an equivalence of categories*

$$\mathcal{B}(G, U_0) \simeq Bun(G, \widehat{U}) \times_{Bun(H, \widehat{U})} \mathcal{B}(H, U_0)$$

*Proof.* To prove (i) suppose that  $\mathcal{P}, \mathcal{P}'$  are two principal bundles on  $\widehat{U}$  admitting algebraizations  $P, P'$ , respectively, which we may assume to be defined on the same  $U \subset X$ . Let  $\widehat{\psi} : \mathcal{P} \rightarrow \mathcal{P}'$  be an isomorphism. We need to prove that there exists (perhaps after shrinking  $U$ ) a unique isomorphism  $\psi : P \rightarrow P'$  with completion given by  $\widehat{\psi}$ . Let  $Isom(P, P')$  be the bundle of isomorphisms  $P \rightarrow P'$ . Considering graphs of isomorphisms, we can identify  $Isom(P, P') \simeq G \setminus (P \times_U P')$ . On the other hand,  $P \times_U P'$  is a principal bundle over  $G \times_k G$ . Define a left action of  $G \times_k G$  on  $G$  by  $(g, h) \cdot f = gfh^{-1}$ , then  $G \setminus (P \times_U P') \simeq (P \times_U P')_G$ . Since  $\widehat{\psi}$  gives a section  $\widehat{s}$  of  $Isom(\mathcal{P}, \mathcal{P}')$ , applying Lemma 11 to  $H = G \times_k G$  and  $Y = G$ , we get a unique algebraization  $s : W \rightarrow (P \times_U P')_G \simeq Isom(P, P')|_W$ , which corresponds to the required isomorphism  $\psi$ . This proves (i).

The statement of (ii) for objects holds by Corollary 8 and for morphisms by (i).

For (iii) first observe that the compositions  $\mathcal{B}(G, U_0) \rightarrow \mathcal{B}(H, U_0) \rightarrow Bun(H, \widehat{U})$  and  $\mathcal{B}(G, U_0) \rightarrow Bun(G, \widehat{U}) \rightarrow Bun(H, \widehat{U})$  are canonically isomorphic, therefore one does get a functor

$$\mathcal{B}(G, U_0) \rightarrow Bun(G, \widehat{U}) \times_{Bun(H, \widehat{U})} \mathcal{B}(H, U_0)$$

On objects, this functor is an equivalence if for a  $G$ -bundle  $\mathcal{P}$  on  $\widehat{U}$ , an  $H$ -bundle  $Q$  on  $U \subset X$  and an isomorphism  $\phi : \mathcal{P}_H \simeq \widehat{Q}$ , there exists an open subset  $W \subset U$  with  $W \cap X_0 = U_0$ , a  $G$ -bundle  $P$  on  $W$  and isomorphisms  $\widehat{P} \simeq \mathcal{P}$  and  $P_H \simeq Q|_W$  which induce  $\phi$  in a natural way. This is equivalent to finding an algebraization of the section  $\widehat{s} : \widehat{U} \rightarrow \widehat{Q}_{H/G}$  induced by  $\phi$ , which was done in the proof of Proposition 10. On morphisms, without loss of generality it suffices to consider two  $G$ -bundles  $P, P'$  defined on the same open set  $U$ , and isomorphisms  $\psi : P_H \simeq P'_H$ ,  $\widehat{\phi} : \widehat{P} \rightarrow \widehat{P}'$  which have the same image in  $Bun(H, \widehat{U})$ . We need to show that there exists a unique isomorphism  $\phi : P \rightarrow P'$  inducing  $\widehat{\phi}$  and  $\psi$  in the natural sense. But by (i) there exists a unique  $\phi$  with completion equal to  $\widehat{\phi}$ . Since by assumption the isomorphisms  $\psi' = \phi_H$  and  $\psi$  are equal after completion,  $\psi' = \psi$  by part (i). This finishes the proof.  $\square$

## References

- [Ar] Artin, M.: Versal deformations and algebraic stacks, *Invent. Math.* **27** (1974), 165–189.
- [BFG] Braverman, A.; Finkelberg M.; Gaitsgory D.: Uhlenbeck spaces via affine Lie algebras, *The unity of mathematics*, 17–135, Progr. Math., 244, *Birkhäuser, Boston, MA*, 2006.
- [DK] Donaldson, S. K.; Kronheimer, P. B.: “The geometry of four-manifolds”. Oxford Mathematical Monographs. *The Clarendon Press, Oxford University Press, New York*, 1990.
- [EGA I] Grothendieck, A.: EGA I “Le langage des schémas”. *Inst. Hautes Études Sci. Publ. Math.* No. 4 1960.
- [EGA III] Grothendieck, A.: EGA III “Étude cohomologique des faisceaux cohérents, première partie”. *Inst. Hautes Études Sci. Publ. Math.* No. 11 1961.
- [FGK] Finkelberg, M.; Gaitsgory, D.; Kuznetsov, A.: Uhlenbeck spaces for  $A^2$  and the affine Lie algebra  $\hat{sl}_n$ . *Publ. Res. Inst. Math. Sci.* **39** (2003), no. 4, 721–766.
- [Ha1] Haboush, W.J.: Homogeneous vector bundles and reductive subgroups of reductive algebraic groups, *Amer. J. of Math.* **100** (1978), no. 6, 1123–1137.
- [Ha2] Hartshorne, R.: “Residues and Duality”. Lecture Notes in Mathematics, No. 20 *Springer-Verlag, Berlin-New York*, 1966.
- [Li] Li, J.: Algebraic geometric interpretation of Donaldson’s polynomial invariants, *Journal of differential geometry* **37** (1993), 417–466.
- [Lu] Lurie, J.: PhD Thesis available through the MIT Library website.
- [R] Raynaud, M.: “Théorèmes de Lefschetz en cohomologie cohérente et en cohomologie étale”. *Mém. Soc. Math. France*, **41**, Paris, 1975.
- [SGA2] Grothendieck, A.: SGA 2 “Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux”. Revised reprint of the 1968 French original. *Documents Mathématiques (Paris)*, 4. Société Mathématique de France, Paris, 2005.

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